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Generalized Vaidya Metrics†

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Abstract

We investigate criteria under which one may construct the energy tensor of a null radiation field from an algebraically special vacuum metric. The field bears the same relationship to the original metric as does Vaidya's to Schwarzschild's. As an example we generate a class of null radiation fields from a class of vacuum metrics without symmetry discovered by Robinson and Robinson.

1. *Introduction and Field Equations*

This paper establishes a method for constructing the energy tensor of a null radiation field from the metric of an algebraically special vacuum gravitational field. We consider a covariant generalization of the relationship between the Schwarzschild metric and the Vaidya (1951) metric and derive integrability conditions for the resulting field equations. It follows as an interesting consequence of these conditions that the radiation fields with twisting rays are much more severely restricted than those with hypersurface-orthogonal rays: indeed, in the twisting case the field amplitude is determined explicitly by the vacuum metric up to a constant factor. Our motivation stems from the fact that these fields might be representative of radiation being emitted from a spinning mass.

Robinson *et al.* (1969) have shown that if a spacetime admits a null vector k^μ tangent to a nonshearing, diverging congruence of affinely parametrized null geodesic curves then coordinates§

$$x^\mu = \{\zeta, \tilde{\zeta}, \sigma, \rho\} \tag{1.1}$$

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§ Indices range over 1, 2, 3, 4; a comma denotes partial differentiation; square brackets denote antisymmetrization over the enclosed indices; the tilde denotes complex conjugation.

may be chosen such that in a vacuum the *main* gravitational field equations

$$k_{[\mu}R_{\nu][\sigma}k_{\tau]} = 0 \tag{1.2}$$

have the solution

$$ds^2 = 2P^2\, d\zeta\, d\tilde{\zeta} + 2d\Sigma(d\rho + Z\, d\zeta + \tilde{Z}\, d\tilde{\zeta} + S\, d\Sigma) \tag{1.3a}$$

$$d\Sigma := a(b\, d\zeta + \tilde{b}\, d\tilde{\zeta} + d\sigma) = k_\mu\, dx^\mu \tag{1.3b}$$

$$P^2 := \exp(2u)(\rho^2 + \Omega^2) \tag{1.3c}$$

$$S := \rho u_3 - \tfrac{1}{4}(K + \tilde{K}) + (\rho m + \Omega M)/(\rho^2 + \Omega^2) \tag{1.3d}$$

$$Z := \rho\Lambda - i(\Omega_1 + \Lambda\Omega) \tag{1.3e}$$

$$\Omega := \tfrac{1}{2} ia \exp(-2u)(\tilde{b}_1 - b_2) \tag{1.3f}$$

$$\Lambda := a^{-1} a_1 - ab_3 \tag{1.3g}$$

$$K := 2\exp(-2u) L_2 \tag{1.3h}$$

$$L := \Lambda - u_1 \tag{1.3i}$$

$$M := \tfrac{1}{2} i \exp(-3u)(U_{1122} - U_{2211}) \tag{1.3j}$$

where a, b, u, m, and U are functions of ζ, $\tilde{\zeta}$, and σ subject only to

$$a \neq 0, \qquad U_3 = \exp(-u) \tag{1.4}$$

We use the notation

$$df = f_1\, d\zeta + f_2\, d\tilde{\zeta} + f_3\, d\Sigma \tag{1.5}$$

for any function $f(\zeta, \tilde{\zeta}, \sigma)$.

Robinson & Robinson (1969) note that in a vacuum the remaining *subsidiary* field equations are equivalent to the vanishing of a form dC:

$$dC := [\rho^{-3}(m - iM)]_{/1}\, d\zeta + [\rho^{-3}(m + iM)]_{/2}\, d\tilde{\zeta} + \{[\rho^{-3}(m + iM)]_{/3} + \rho^{-4}\exp(-4u) I\}\, \rho\, d\Sigma = 0 \tag{1.6}$$

where we define

$$I := J_{22} + 2\tilde{L}J_2 \tag{1.7}$$

$$J := L_1 + L^2 \tag{1.8}$$

and use the notation

$$df = f_{/1}\, d\zeta + f_{/2}\, d\tilde{\zeta} + f_{/3}\, \rho\, d\Sigma + f_{/4}\, dW \tag{1.9}$$

$$dW := \Lambda\, d\zeta + \tilde{\Lambda}\, d\tilde{\zeta} + u_3\, d\Sigma + \rho^{-1}\, d\rho \tag{1.10}$$

for any function $f(\zeta, \tilde{\zeta}, \sigma, \rho)$.

We ask for conditions necessary and sufficient to ensure that if $g_{\mu\nu}$ is a solution of the vacuum field equations (1.2) and (1.6), then there exists a function $H(\zeta, \tilde{\zeta}, \sigma, \rho)$ such that the substitution

$$g_{\mu\nu} \to g_{\mu\nu} + 2H\rho^4(\rho^2 + \Omega^2)^{-1} k_\mu k_\nu \tag{1.11}$$

yields a solution of Einstein's equations with an energy tensor proportional to $k_\mu k_\nu$. Calculation shows that the resulting energy tensor is associated with the *flux* of H:

$$R_{\mu\nu} \to R_{\mu\nu} + H_{/3}\, \rho^4(\rho^2 + \Omega^2)^{-1} k_\mu k_\nu \tag{1.12}$$

To elucidate these conditions we remark that substitution (1.11) may be expressed equivalently by

$$m \to m + H\rho^3 \tag{1.13}$$

which in turn implies that

$$dC \to dC + H_{/1}\, d\zeta + H_{/2}\, d\tilde{\zeta} + H_{/3}\, \rho\, d\Sigma \tag{1.14}$$

We find that the main equations are satisfied only if

$$H_{/4} = -3H \tag{1.15}$$

and that the subsidiary equations may be written

$$dC = H_{/3}\, \rho\, d\Sigma \tag{1.16}$$

We thereby obtain

$$H_{/1} = 0, \qquad H_{/2} = 0 \tag{1.17}$$

using equation (1.14).

2. *Integrability Conditions*

The commutation relation

$$f_{/[12]} = -\tfrac{1}{2} \exp(2u)\, [2i\rho\Omega f_{/3} + \tfrac{1}{2}(K - \tilde{K}) f_{/4}] \tag{2.1}$$

for an arbitrary function $f(\zeta, \tilde{\zeta}, \sigma, \rho)$ provides an integrability condition for the field equations (1.15) and (1.17). From the Jacobi identity for the derivatives defined by equation (1.9) follow the relations

$$(\rho^{-1}\Omega)_{/3} = \tfrac{1}{4} i \rho^{-2}(K - \tilde{K}) \tag{2.2}$$

$$\tfrac{1}{2}[\rho^{-2}(K - \tilde{K})]_{/3} = (\rho^{-1}\tilde{L}_3)_{/1} - (\rho^{-1}L_3)_{/2} \tag{2.3}$$

Substituting Jacobi relation (2.2) into commutation relation (2.1), and using the field equations, we integrate, providing that Ω does not vanish identically, to obtain

$$H = h(\zeta, \tilde{\zeta})\, [\exp(2u)\, \Omega\rho]^{-3} \tag{2.4}$$

Defining a function $G(\zeta, \tilde{\zeta}, \sigma)$ by

$$G := [\ln(\rho^{-1}\Omega)]_{/1} - 2L \tag{2.5}$$

we use the commutation relation to establish that

$$G_2 = \tilde{G}_1 \tag{2.6}$$

From equations (1.17) it follows as a second integrability condition that

$$G = G(\zeta, \tilde{\zeta}) \tag{2.7}$$

Equations (1.17) together with the compatibility condition (2.6) then explicitly determine h to be

$$h = \eta \exp\left(3 \int G\, d\zeta\right) \tag{2.8}$$

where η is a constant.

3. *Special Solutions*

In order that substitution (1.11) results in a non-trivial radiation field it is necessary that Ω satisfy one of the conditions:

$$\Omega = 0; \qquad (\rho^{-1}\Omega)_{/3} \neq 0 \tag{3.1}$$

In the first instance k^μ is hypersurface-orthogonal; the radiation field belongs to a class mentioned by Robinson & Trautman (1962). In this case, in suitable coordinates, $H\rho^3$ is a disposable function of σ. The Vaidya (1951) metric constitutes an example of such a field. Among the excluded vacuum fields satisfying neither of equations (3.1) we find stationary fields with twist, such as those of Kerr (1963), Newman *et al.* (1963), and Robinson *et al.* (1969).

To construct an example of a field in accordance with the second of equations (3.1) we consider a class of vacuum metrics without symmetry discovered by Robinson & Robinson (1969), characterized by

$$L_3 = 0, \qquad (\rho^{-1}\Omega)_{/3} \neq 0 \tag{3.2}$$

Imposing as coordinate conditions

$$a = 1, \qquad u_3 = 0 \tag{3.3}$$

we then obtain

$$\Omega_{,33} = 0 \tag{3.4}$$

from the Jacobi relations (2.2) and (2.3). The coordinate conditions determine σ up to the product of a shift in origin,

$$x^\mu \to \{\zeta, \tilde{\zeta}, \sigma + \delta(\zeta\tilde{\zeta}), \rho\} \tag{3.5}$$

and a change in scale,

$$x^\mu \to \{\zeta, \tilde{\zeta}, \sigma/\gamma(\zeta, \tilde{\zeta}), \rho\gamma(\zeta, \tilde{\zeta})\} \tag{3.6}$$

Thus, fixing the origin of σ by writing

$$\Omega = \tfrac{1}{2} i \exp(-2u)(L_{,2} - \tilde{L}_{,1})\,\sigma \tag{3.7}$$

it becomes apparent that integrability condition (2.7) is satisfied. Equation (2.8) then results in

$$H = \alpha(\sigma\rho)^{-3} \tag{3.8}$$

where α is a constant; this expression is manifestly invariant under changes in scale.

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